LLNL-TR-423102

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Simple Model of the (alpha)(omega) Dynamo: Self-Excited Spheromaks

T. K. Fowler

February 3, 2010

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

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January 25, 2010

Abstract

The astrophysical $\alpha\omega$ dynamo converting angular momentum to magnetic energy can be interpreted as a self-excited Faraday dynamo together with magnetic relaxation coupling the dynamo poloidal field to the toroidal field produced by dynamo currents. Since both toroidal and poloidal fields are involved, the system can be modeled as helicity creation and transport, in a spheromak plasma configuration in quasi-equilibrium on the time scale of changes in magnetic energy. Neutral beams or plasma gun injection across field lines could create self-excited spheromaks in the laboratory.

Introduction: The $\alpha\omega$ Dynamo

E. N. Parker first proposed introducing an $\alpha\omega$ dynamo into Faraday's Law [1, 2], which we choose to write as (MKS) [3]:

$$\nabla\phi + \partial\mathbf{A}/\partial t = -\mathbf{E} = \mathbf{v} \times \mathbf{B} - \mathbf{R} \tag{1}$$

Here $\mathbf{R}$ represents resistive losses and also turbulent transport. For axisymmetry ($\mathbf{v} = \mathbf{v}_\phi$, $\nabla_\phi\phi = 0$), $\mathbf{v} \times \mathbf{B}$ in Eq. (1) drives toroidal flux $\propto \mathbf{A}_{POL}$ (giving B_ϕ) but it does not directly drive poloidal flux $\propto \mathbf{A}_\phi$. To obtain a self-excited dynamo, Parker added an ad hoc term $\alpha\mathbf{B}_\phi$ which couples poloidal and toroidal fluxes. Parker attributed α to pre-existing turbulence causing the growth of a coherent dynamo field from a very weak seed field around an accretion site.

The $\alpha\omega$ Dynamo as Helicity Injection

Since the $\alpha\omega$ self-excited dynamo involves a magnetic field with both toroidal and poloidal fields, the field has helicity. We define helicity as $K = \int d\mathbf{x}\, \mathbf{A}\cdot (\nabla \times \mathbf{A})$, with no gauge ambiguity if $\mathbf{A}$ only represents magnetic fluxes [5]. Using Eq. (1), we obtain:

$$dKdt = \int d\mathbf{x}\, \nabla\cdot(\partial \mathbf{A}/\partial t \times \mathbf{A}) - 2\int d\mathbf{x}\, \nabla\phi\cdot\mathbf{B} - 2\int d\mathbf{x}\, \mathbf{R}\cdot\mathbf{B} \qquad (2)$$

The first term on the right represents inductive current drive in fusion tokamaks and reversed-field pinches (RFP's). The second term represents electrostatic generation of helicity, while the term with **R** represents resistive losses, and also turbulent transport of helicity, which however, conserves helicity for most processes [3, 4]. Eq. (2) has been invoked as a unified description of helical fields, by Boozer to describe current drive in fusion tokamaks, RFP's and spheromaks [5], and more recently by Blackman and Ji to unify these fusion applications with self-excited dynamos in astrophysics [6].

That Eq. (2) can describe an $\alpha\omega$ dynamo driven by momentum follows from the coupling of helicity to momentum injection, as follows. Since most of the helicity is in the mean field $\mathbf{B}_o$ [3, 4], Eq. (2) is equivalent to the mean field energy equation obtained from the symmetric average of **B** dotted into Faraday's Law, giving:

$$\partial(B_o^2/2\mu_o)/\partial t = \mu_o^{-1}\nabla\cdot\langle(\mathbf{v} \times \mathbf{B}) \times \mathbf{B}\rangle + \langle\mathbf{j}\cdot(\mathbf{v} \times \mathbf{B})\rangle - (B_o^2/2\mu_o)/\tau_\Omega \qquad (3)$$

where τ_Ω is the ohmic decay time and we omit perturbations in B^2. The corresponding kinetic equation is found by dotting **v** into the momentum equation, giving:

$$\rho\partial(v_o^2/2)/\partial t = \mathbf{v}_o\cdot\mathbf{F} - \langle\mathbf{j}\cdot(\mathbf{v} \times \mathbf{B})\rangle - \rho(v_o^2/2)/\tau_M + \mathbf{v}_o\cdot\mathbf{S} \qquad (4)$$

$$\mathbf{F} = \{\rho(v_{o\phi}^2/r) - \rho\nabla V_G - \nabla p\} \qquad (5)$$

with symmetrized volume momentum source **S** and force **F**; momentum confinement time τ_M; pressure p; gravitational potential V_G; and we use $\mathbf{v}\cdot(\mathbf{j} \times \mathbf{B}) = -\mathbf{j}\cdot(\mathbf{v} \times \mathbf{B})$. Adding Eqs. (3) and (4) introduces the momentum power source $\mathbf{v}_o\cdot\mathbf{S}$ into Eq. (3); or equivalently, a drive term in Eq. (2) given approximately by $(2\mu_o/\lambda)\int d\mathbf{x}\, \mathbf{v}_o\cdot\mathbf{S}$ where λ is the lowest eigenvalue of $\nabla \times \mathbf{B} = \lambda\mathbf{B}$ defined on the volume where momentum is injected.

The $\alpha\omega$ Dynamo as an Internal Electrostatic "Gun"

Ignoring induction, we can write Eq. (2) as it usually appears in spheromak literature describing electrostatic helicity injection [4, 5]:

$$dK/dt = 2V\Psi_{GEN} - K/\tau_K \tag{6}$$

where Ψ_{GEN} is poloidal flux in a magnetized plasma gun with electrostatic voltage V applied across this flux, and τ_K represents the last term in Eq. (2), with $\tau_K \approx \tau_\Omega$ as noted above (or loss by cosmic rays as discussed below). In the laboratory, V (coming from ϕ in Eq. (2)) has usually been applied by a capacitor bank, though conceptually V could be supplied by a Faraday disk giving $V = \Omega\Psi_{GEN}$ for rotation frequency Ω, and the entire Faraday gun assembly could face the plasma directly via a hole in the vessel wall [7].

Taking a clue from Siemen's 19th century invention of the self-excited dynamo [2], one might use the dynamo current itself to produce the Faraday dynamo bias field, giving also a toroidal field due to the current, hence helicity. Replacing the rotating disk by a plasma with non-uniform rotation yields a self-excited dynamo inside the plasma. This is the Parker $\alpha\omega$ dynamo, in which, given even an infinitesimal seed field, angular momentum alone creates helicity in the plasma. A self-excited $V = \Omega\Psi_{POL}$ arises from ϕ produced by charge density $\propto\nabla\cdot(\mathbf{v}\times\mathbf{B})$ [8]. Helicity injection is then given by:

$$dK/dt = 2\Omega\Psi_{POL}^{2} - K/\tau_K \tag{7}$$

where $K \approx 2\Psi_{POL}\Psi_{TOR}$.

Role of Magnetic Relaxation in Self-Excitation

We define magnetic relaxation broadly as any mechanism coupling Ψ_{POL} and Ψ_{TOR} so that creating one produces the other also. By Eq. (7), K could grow exponentially if magnetic relaxation produces $\Psi_{POL} \propto \Psi_{TOR}$, analogous to Parker's α. Magnetic relaxation in laboratory RFP's and spheromaks has been attributed to MHD tearing modes; explicit α processes have been proposed to explain astrophysical dynamos [2, 9]; and both laboratory experience and simulations of experiments suggest that magnetic relaxation in the hot plasmas of astrophysics and fusion research is probably ubiquitous unless it is deliberately suppressed (as in tokamaks). Thus we will assume that some mechanism of magnetic relaxation coupling toroidal and poloidal fluxes can be available in fusion experiments, and generally would be in astrophysical applications where there is ample evidence of strong magnetic fields around accretion sites [8].

Magnetic relaxation by turbulence has not always been useful in fusion research [4], and MHD turbulence might kill the dynamo [9]. The mechanism of Ref. [9] is thought to avoid turbulence, as could injecting both components of **v** in Eq. (1) in a fusion device. The likely circumstance that accretion yields only angular momentum requires a persisting α, together with the toroidal mean-field equation found by taking the toroidal component of the curl of Eq. (1):

$$\partial B_\phi/\partial t = r(\partial\Omega/\partial r)B_r + r(\partial\Omega/\partial z)B_z = r\nabla\Omega\cdot\mathbf{B}_{POL} \qquad (8)$$

Thus $\mathbf{B}_{POL}$ drives B_ϕ, by sheared rotation [2, 9], while localization in z of the dynamo EMF ($\propto\Omega$) drives current along field lines exiting the spinning dynamo region.

MHD problems can be diagnosed in simulations by monitoring growth of local cross helicity $\mathbf{v}\cdot\mathbf{B}$ which kills the dynamo if **v** fully aligns with **B** [10].

<u>Simplified $\alpha\omega$ Dynamo</u>

Besides assuming some mechanism coupling poloidal and toroidal fluxes (which we have defined as magnetic relaxation, not necessarily turbulent), dynamo physics can be further simplified when helicity injection is slow on timescales establishing quasi-equilibrium via the momentum equation. Then we can replace dynamics in Eq. (4) by a sequence of quasi-equilibria. Doing this and using Eq. (7) with $K \approx \Psi_{POL}\Psi_{TOR}$, we obtain as the dynamo model:

$$dK/dt = \Omega K(\Psi_{POL}/\Psi_{TOR}) - K/\tau_K \qquad (9)$$

$$0 = \mathbf{j} \times \mathbf{B} + \{\rho(v_{o\phi}{}^2/r) - \rho\nabla V_G - \nabla p\} \qquad (10)$$

with the caveat that angular momentum input via Ω averaged over the dynamo cannot exceed the net input of momentum against momentum losses.

Application to Accretion Disks

An example astrophysical application is the calculation of parameters for jet/radiolobes produced by accretion around supermassive black holes. Refs. [11] and [12] use Eq. (10) to approximate the magnetic structure, giving simply $\Psi_{TOR} = (L/a)(\ln R/a)\Psi_{POL}$ with dynamo radius a, jet length L >> a, and radiolobe radius $R \approx L$ (confined by ambient pressure). Since the system is relatively isolated, we can ignore both momentum losses and helicity losses, giving $dL/dt = (a\Omega/\ln R/a)$, which is much less than Alfven times for this problem, hence justifying dynamics as a series of quasi-equilibria as in Eq. (10). This dL/dt can be related to the black hole accretion rate dM/dt using equilibrium dimensions scaled on the Swarzchild radius $\propto$ the black hole mass M at any chosen time. These simple results prove to be a sufficient framework to deduce that the jet/radiolobe becomes a cosmic ray accelerator, and from this the actual loss of poloidal flux, hence helicity, taken away by the cosmic rays [11].

The essential feature is the self-excited dynamo, which then dominates the dynamics. By Eq. (9), this requires, first, that the dynamo is self-excited to the level required to produce jet-like equilibria by Eq. (10) [11]. The dynamo itself is just that part of the equilibrium consisting of a rotating plasma confined by gravity, the kinetic angular momentum of the dynamo ions being converted to canonical angular momentum of ion current flowing up the jet column [13]. The dynamo boundary is located in the disk corona where dynamo current can no longer be contained by gravity [11, 13]. Since the coronal field would be quasi-force free with $\Psi_{TOR} \approx \Psi_{POL}$, K would grow exponentially by Eq. (9) until helicity spills out of the dynamo as a jet of current growing in length as M grows, at constant I equal to the current at the threshold of gravitational confinement of the dynamo current [11]. One could extend the coronal equilibrium model of Ref. [14] to the interior if, for example, the region interior to the corona is dominated by radiation pressure near the Eddington limit [8, 12].

As this example suggests, magnetic relaxation essential to the self-excited dynamo does not necessarily mean relaxation to a state of minimum magnetic energy at constant helicity. The extent of relaxation in the region of helicity generation is in part controlled by the momentum source [4], and the expanding radiolobe is only a little bit relaxed, giving a magnetic energy and helicity dominated by the lobe vacuum field due to the jet current deep inside the lobe (hence the factor lnR/a above) [11].

Application to Fusion Reactors

Up to now, applications of helicity injection in fusion research have employed external helicity sources, as in the case of electrostatic gun injection mentioned above. In RFP's and spheromaks, magnetic relaxation is involved to satisfy equilibrium, giving excessive heat loss [4]. Magnetic relaxation is not required for tokamaks employing neutral beam injection (NBI) just to produce toroidal current. NBI current could also build up current in spheromaks, possibly maintaining a stable current profile [10]. The NBI current drive in tokamaks is thought to inject current directly (the Ohkawa current parallel to **B**). Though this can give a fusion power gain Q > 10 in tokamaks, and also spheromaks [10], greater efficiency is highly desirable. Greater efficiency could be achieved using NBI-injected momentum to drive a self-excited dynamo in spheromaks. The efficiency can be written as:

$$I/P = (\tau_{DYN}/\Psi_{POL}) \tag{13}$$

where $\tau_{DYN} = (\tau_M/\beta_M) \leq \tau_\Omega$ with momentum lifetime τ_M from Eq. (4) (by viscosity or transport to the walls) and $\beta_M = (\rho v_\phi^2 \mu_o/B^2)$ representing energy partition by Eqs. (3) and (4). Rearranging terms gives $\Psi_{POL} = (P/I)\tau_{DYN}$ in volt-seconds. The time τ_{DYN} is unknown, but might be several times the particle containment time, hence > 10 sec for energy confinement times > 1 sec. Energy confinement sufficient for fusion ignition typically requires $\Psi_{POL} \approx RI \approx 100$, perhaps giving an efficiency much greater than the Ohkawa efficiency.

Plasma gun injection across field lines also injects momentum. In a uniform field electric polarization allows the plasma to flow unimpeded, while in a helical field **v** x **B** polarization drives currents that transfer gun momentum to magnetic energy. This internal dynamo effect has been demonstrated in tokamaks [15].

Example

An example velocity field creating a dynamo in 2D without resorting to turbulence can be found by introducing $\mathbf{A} = f(t)\, \mathbf{A}_o\, (\mathbf{x})$ into Eq. (1) with $\nabla\phi = 0$ (fully efficient use of the polarization electric field). This gives:

$$\mathbf{v}_\perp \quad = \quad -(df/dt\, \mathbf{A}_o \times \mathbf{B}_o)/B_o^{\,2} \tag{14}$$

For $\mathbf{A}_o$ giving a finite result, an exponential f would give exponentially increasing helicity in the dynamo until some other limit is reached (e.g. jet ejection from an accretion dynamo).

In practice any **v** giving a component of **v** x **B** parallel to **A** can drive helicity. An example using gun injection to achieve nuclear ignition in a spheromak by ohmic heating alone will be discussed in a separate paper in preparation.

Discussion

We have derived a simplified model of the αω dynamo that unifies its application to astrophysics and to fusion research. The model treats dynamics as a change in helicity, dK/dt by Eq. (9), for magnetic fields evolving as a sequence of equilibria, by Eq. (10). For all applications, persistence of the dynamo depends on the stability of this sequence of equilibria, to avoid excessive heat leakage and to avoid killing the dynamo by MHD turbulence causing **v** to align with **B** [10]. Implications for accretion dynamos in astrophysics are discussed in Ref. [9], and for fusion, in Refs. [12] and [13] and a paper in progress. Otherwise the main difference between the astrophysical dynamos and self-excited spheromak dynamos for fusion concern the mission: in astrophysics, to explain jet/radiolobes as a failure of dynamo plasma confinement in the dynamo; in fusion, to achieve plasma confinement good enough to achieve nuclear ignition in a device smaller than tokamaks.

Gun experiments of interest to fusion may also shed light on cross-field momentum propagation during accretion, for example, polarization effects on "winds" ejected out of a dynamo, discussed in Ref. [16]. Jets penetrating deeply inward might also contribute to fueling a black hole by shedding momentum as current. The range of a jet at which its kinetic energy is dissipated electrically is $\approx w\,\beta_{JET}\cos\theta$ for a jet of width w, angle θ relative to $\mathbf{B}$, and ratio β_{JET} of kinetic to magnetic energy density.

I would like to thank Harry McLean and Bick Hooper for helpful discussions.

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This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.